LECTURE 9

Cost Minimisation

5SSPP217 – Intermediate Microeconomics

Recap: the firm's objective and constraint

The firm's **objective** is to maximise profit, taking input and output prices as given.

Its **constraint** is technology: which combinations of inputs and output are feasible?

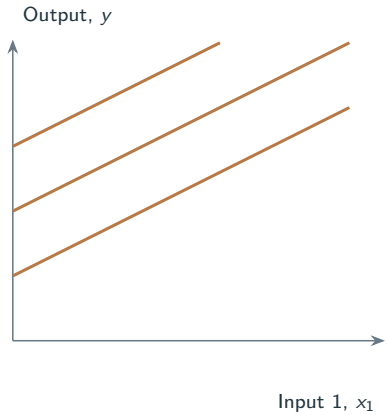
$$\max_{x_1, x_2, y \geq 0} \; py - w_1 x_1 - w_2 x_2 \quad \text{subject to} \quad y \leq f(x_1, x_2).$$

The **production function** $f(x_1, x_2)$ represents technology by giving the maximum output obtainable from each input bundle.

Recap: profit maximisation

In the short run, input 2 is fixed at $\bar{x}_2$. Given prices, the firm chooses input 1 and hence output.

An **isoprofit line** shows input–output combinations with the same profit. Higher parallel lines mean higher profit.



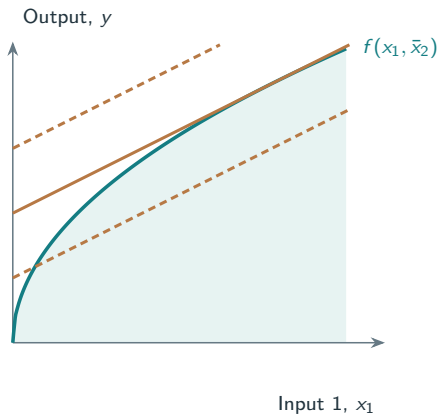
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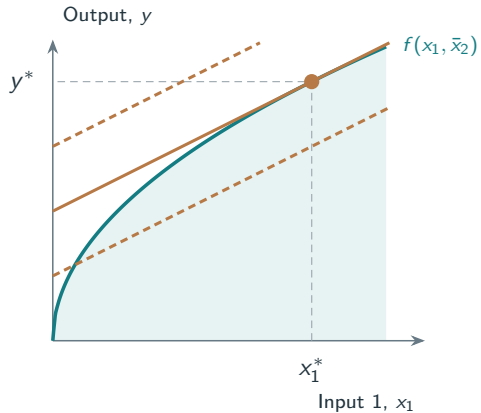
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Technology limits output:

$$y \leq f(x_1, \bar{x}_2).$$

The firm chooses (x_1^*, y^*) on the **highest attainable isoprofit line**.



Production method and output choice

In last Thursday's lecture, we studied how firms chose inputs and hence output to maximise profits.

We can separate this problem into two decisions:

1. How to produce: given a specific output target, what is the combination of inputs with the lowest cost?
2. How much to produce: what is the most profitable level of output?

In this lecture, we will look at the first step.

Choosing the production method

The cost-minimisation problem

What is the cheapest way to produce a given level of output $y > 0$ at given input prices $w_1, w_2 > 0$?

The firm solves the following minimisation problem

$$c(w_1, w_2, y) = \min_{x_1, x_2 \geq 0} \; w_1 x_1 + w_2 x_2 \quad \text{subject to} \quad f(x_1, x_2) = y.$$

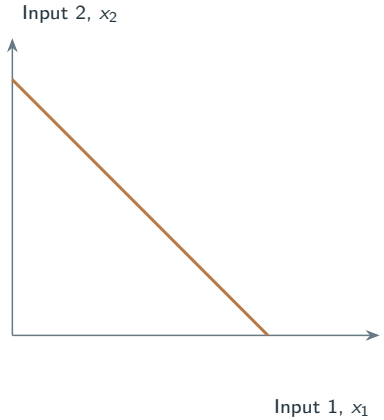
In words: the firm chooses quantities of inputs 1 and 2 to minimise the cost of producing y.

The solution to this problem is $c(w_1, w_2, y)$, which we call the **cost function**: it gives the minimum expenditure needed to produce a specific level of output.

Isocost lines

The **isocost line** contains all input bundles that cost the same amount C:

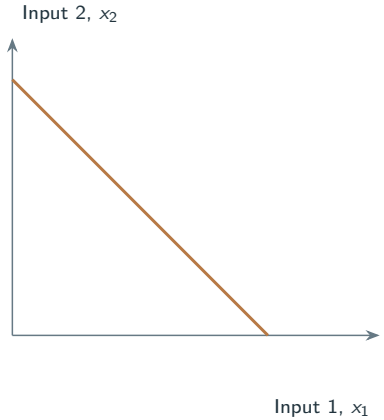
$$w_1 x_1 + w_2 x_2 = C$$



Isocost lines

The **isocost line** contains all input bundles that cost the same amount C:

$$w_2 x_2 = C - w_1 x_1$$

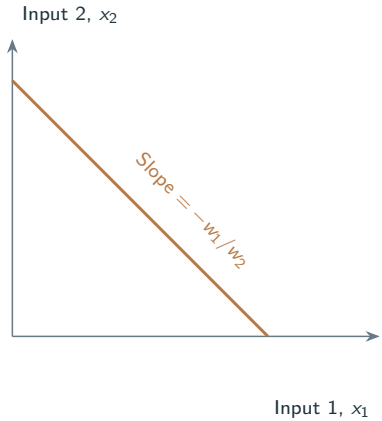


Isocost lines

The **isocost line** contains all input bundles that cost the same amount C:

$$x_2 = \frac{C}{w_2} - \frac{w_1}{w_2} x_1$$

One extra unit of input 1 costs w_1. To keep cost unchanged, the firm must save w_1 on input 2. Since each unit of input 2 costs w_2, it must use w_1/w_2 fewer units of input 2. This is why the isocost line has slope $-w_1/w_2$.



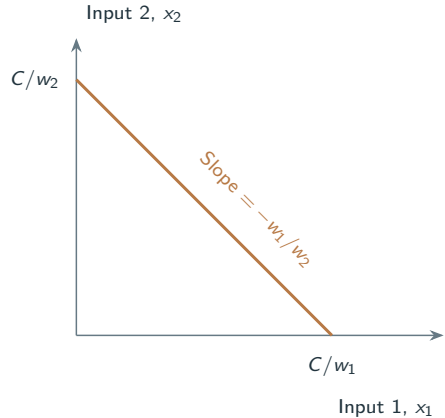
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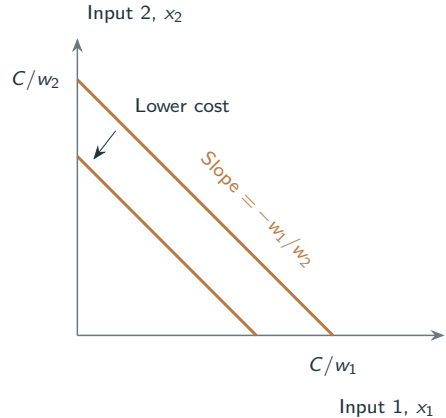
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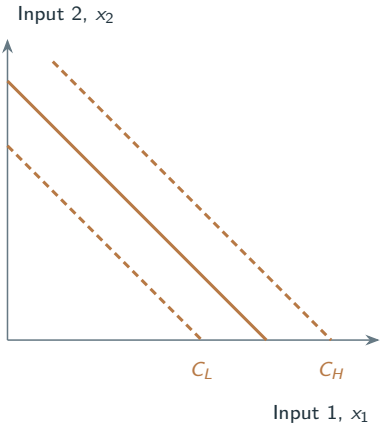
The intercepts are C/w_1 on the horizontal axis and C/w_2 on the vertical axis.

A lower C shifts the line **inwards**: every bundle on it costs less.



The lowest attainable isocost line

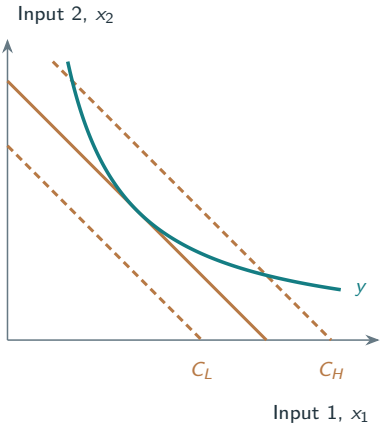
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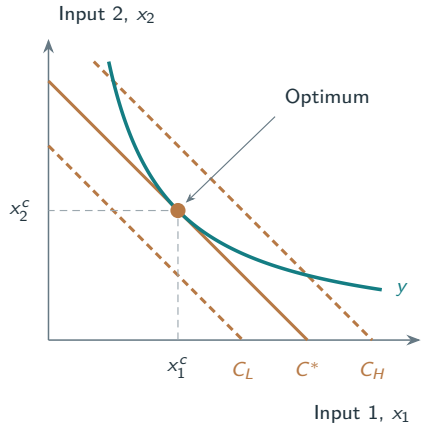
A firm wants the lowest possible cost of producing a given level of output.

What constrains it? **Technology!** The isoquant shows the input bundles that produce the required output.

The firm chooses the **lowest isocost line that reaches the isoquant**. Any lower line cannot produce the target.

At a smooth interior optimum, the isoquant and isocost line have the same slope:

$$TRS = -\frac{MP_1}{MP_2} = -\frac{w_1}{w_2}.$$

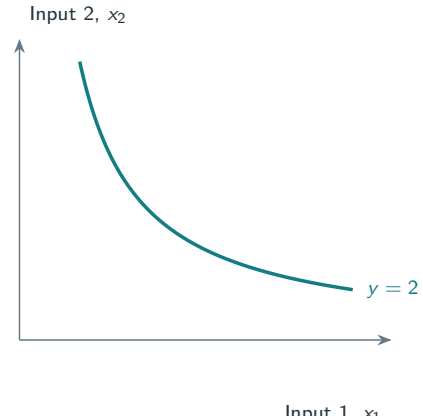


The lowest attainable isocost line: an example

Suppose $f(x_1, x_2) = \sqrt{x_1 x_2}$, the output target is $y = 2$, and $w_1 = w_2 = 1$.

The isoquant and isocost lines are

$$x_1 x_2 = 4, \qquad x_1 + x_2 = C.$$



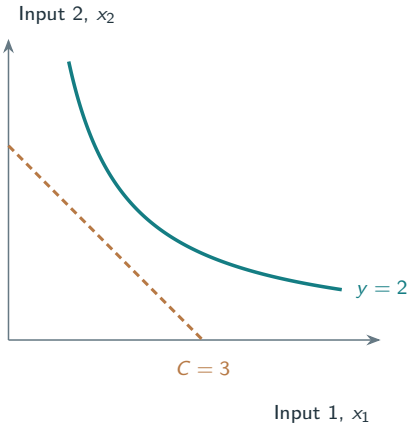
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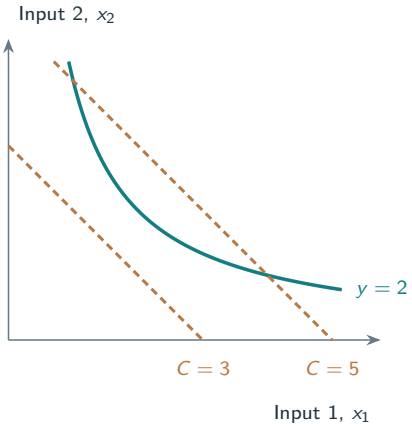
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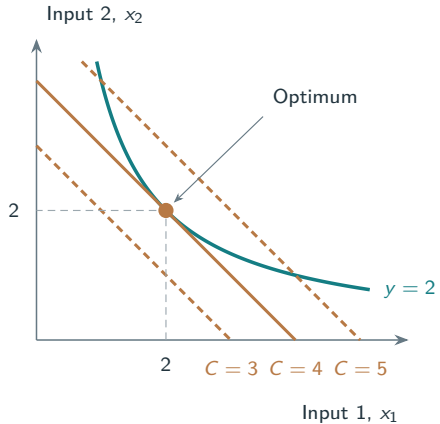
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The minimum is $C = 4$, at $(2, 2)$, where the isocost line is tangent to the isoquant.



Cost minimisation and consumer choice

The diagrams resemble those for consumer choice, but the objective and constraint switch roles.

	Utility maximisation	**Cost minimisation**
Constraint	Given budget line	Given output isoquant
Objective	Reach the highest attainable indifference curve	Reach the lowest attainable isocost line

The consumer faces a fixed budget and seeks higher utility.

The firm holds output fixed and seeks lower expenditure.

The Lagrangian and interior conditions

If there is a smooth interior solution, we can solve the problem of cost minimisation by using the **Lagrangian method**.

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda \big[y - f(x_1, x_2) \big].$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda MP_1 = 0 \quad \Longrightarrow \quad w_1 = \lambda MP_1,$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \lambda MP_2 = 0 \quad \Longrightarrow \quad w_2 = \lambda MP_2,$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = y - f(x_1, x_2) = 0.$$

Tangency and marginal product per pound

Dividing the two input conditions gives

$$\frac{MP_1}{MP_2} = \frac{w_1}{w_2} \quad \Longleftrightarrow \quad TRS = -\frac{w_1}{w_2}.$$

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$$\frac{MP_1}{w_1} = \frac{MP_2}{w_2}.$$

The last pound spent on either input produces the same extra output.

If one input produces more extra output per pound, the firm can reduce costs by using more of it and less of the other input, while keeping output fixed.

Conditional factor demands

Conditional factor demands

The levels of input 1 and 2 that solve the cost minimisation problem, $x_1^c(w_1, w_2, y)$ and $x_2^c(w_1, w_2, y)$, are **conditional factor demands**: the cheapest input quantities for a **given output target**.

$$c(w_1, w_2, y) = w_1 x_1^c(w_1, w_2, y) + w_2 x_2^c(w_1, w_2, y).$$

Cobb–Douglas: solving for the input quantities

Return to $f(x_1, x_2) = \sqrt{x_1 x_2}$. For $y > 0$ and $w_1, w_2 > 0$, the optimum is given by

$$\frac{MP_1}{MP_2} = \frac{\frac{1}{2}\sqrt{x_2/x_1}}{\frac{1}{2}\sqrt{x_1/x_2}} = \frac{x_2}{x_1} = \frac{w_1}{w_2} \quad \Longrightarrow \quad x_2 = \frac{w_1}{w_2} x_1.$$

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Combining this with the output constraint gives us:

$$y^2 = x_1 x_2 = \frac{w_1}{w_2} x_1^2 \quad \Longrightarrow \quad x_1^c = y\sqrt{\frac{w_2}{w_1}}.$$

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Then

$$x_2^c = \frac{w_1}{w_2} x_1^c = y\sqrt{\frac{w_1}{w_2}}.$$

Cobb–Douglas: the cost function

The next step is to insert the conditional factor demands into the cost function:

$$\begin{aligned} c(w_1, w_2, y) &= w_1 x_1 + w_2 x_2 \\ &= w_1 y \sqrt{\frac{w_2}{w_1}} + w_2 y \sqrt{\frac{w_1}{w_2}} \\ &= y\sqrt{w_1 w_2} + y\sqrt{w_1 w_2} \\ &= 2y\sqrt{w_1 w_2}. \end{aligned}$$

For $y = 2$ and $w_1 = w_2 = 1$, the firm chooses $(2, 2)$ at a cost of £4.

Perfect substitutes: a corner or many solutions

Suppose that the firm faces a technology where inputs are perfect substitutes:

$$f(x_1, x_2) = x_1 + x_2.$$

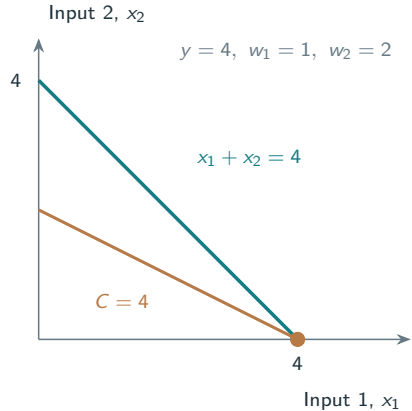
The firm will simply demand whichever input is cheaper.

$$(x_1^c, x_2^c) = \begin{cases} (y, 0), & w_1 < w_2, \\ (0, y), & w_2 < w_1. \end{cases}$$

The cost function can be written as

$$c(w_1, w_2, y) = \min\{w_1, w_2\}\, y.$$

If $w_1 = w_2$, every bundle with $x_1 + x_2 = y$ is optimal.



Perfect complements: a kink

Suppose that the firm faces a technology where inputs are perfect complements:

$$f(x_1, x_2) = \min\{x_1, x_2\}.$$

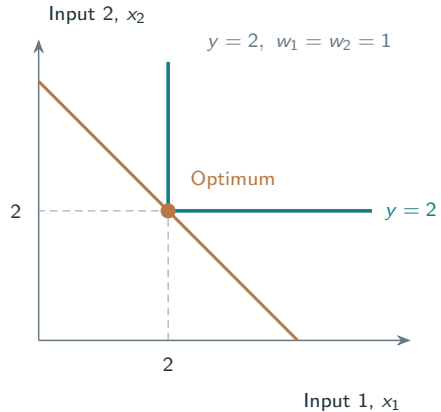
When both inputs are used in equal quantities, increasing just one produces no additional output.

$$x_1^c = x_2^c = y$$

The cost function can be written as

$$c(w_1, w_2, y) = (w_1 + w_2)y$$

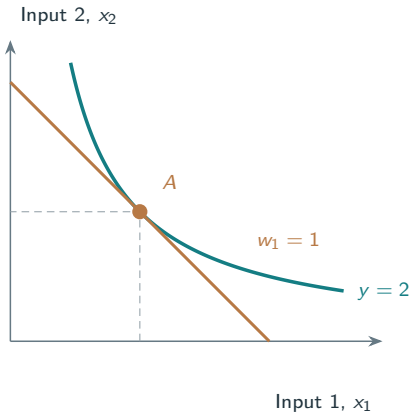
The optimum is at the **kink**. There is no unique isoquant slope there, so the tangency condition does not apply.



An increase in the price of input 1, w_1

Suppose that technology is given by $f = \sqrt{x_1 x_2}$. Hold $y = 2$ and $w_2 = 1$ fixed, and suppose that w_1 rises from 1 to 4.

$$x_1^c = y\sqrt{w_2/w_1}, \qquad x_2^c = y\sqrt{w_1/w_2}.$$



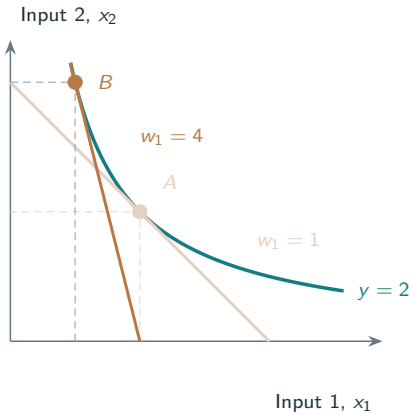
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$$x_1^c = y\sqrt{w_2/w_1}, \qquad x_2^c = y\sqrt{w_1/w_2}.$$

The optimum moves from $A = (2, 2)$ to $B = (1, 4)$. The isocost slope changes from -1 to -4.

The firm substitutes away from input 1.



Output and the adjustment horizon

Constant returns and average cost

The **average cost function** measures cost per unit of output:

$$AC(w_1, w_2, y) = \frac{c(w_1, w_2, y)}{y}, \qquad y > 0.$$

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What are average costs for a firm with **constant returns to scale**?

At fixed input prices, scaling all inputs by t scales output and minimum cost by t:

$$c(w_1, w_2, ty) = t\, c(w_1, w_2, y).$$

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$$c(w_1, w_2, ty) = t\, c(w_1, w_2, y).$$

Set $t = 1/y$. Then

$$c(w_1, w_2, 1) = \frac{c(w_1, w_2, y)}{y} = AC(w_1, w_2, y).$$

Average cost equals the **unit cost function**, $c(w_1, w_2, 1)$, and is independent of output.

Decreasing returns and average cost

Now consider **decreasing returns to scale**, with input prices fixed and all inputs adjustable.

Doubling every input produces **less** than twice the output. At unchanged input proportions, doubling output requires more than doubling every input and its cost.

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Even with the cheapest input mix, costs rise more than proportionately:

$$c(w_1, w_2, 2y) > 2\, c(w_1, w_2, y).$$

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Even with the cheapest input mix, costs rise more than proportionately:

$$c(w_1, w_2, 2y) > 2\, c(w_1, w_2, y).$$

Dividing by the new output level, $2y$, gives

$$AC(w_1, w_2, 2y) = \frac{c(w_1, w_2, 2y)}{2y} > \frac{c(w_1, w_2, y)}{y} = AC(w_1, w_2, y).$$

With decreasing returns, average cost **increases** with output.

Returns to scale, total cost and average cost

Hold input prices fixed, allow all inputs to vary, and compare y with ty, where $t > 1$.

Returns to scale	**Total cost**	**Average cost**
Constant	$c(ty) = t\,c(y)$	$AC(ty) = AC(y)$
Increasing	$c(ty) < t\,c(y)$	$AC(ty) < AC(y)$
Decreasing	$c(ty) > t\,c(y)$	$AC(ty) > AC(y)$

Short-run and long-run cost minimisation

In the **short run**, input 2 is fixed at $\bar{x}_2$ and the firm can only adjust input 1:

$$c_s(w_1, w_2, y; \bar{x}_2) = \min_{x_1 \geq 0} \{ w_1 x_1 + w_2 \bar{x}_2 : f(x_1, \bar{x}_2) = y \}.$$

In the **long run**, the firm can adjust both inputs. The firm can still choose the short-run bundle, but may find a cheaper one:

$$c(w_1, w_2, y) \leq c_s(w_1, w_2, y; \bar{x}_2).$$

Cost minimisation in the short run: an example

For $f = \sqrt{x_1 x_2}$, let $w_1 = w_2 = 1$ and fix $\bar{x}_2 = 4$. To produce y,

$$y = \sqrt{4x_1} = 2\sqrt{x_1} \quad \Longrightarrow \quad x_1^s = \frac{y^2}{4}.$$

Short-run and long-run costs are therefore

$$c_s(y; 4) = \frac{y^2}{4} + 4, \qquad c(y) = 2y \quad \text{(slide 13)}.$$

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At $y = 2$, the short-run bundle is $(1, 4)$ and costs £5.

In the long run, the firm can choose $(2, 2)$, which costs £4.

At $y = 4$, both problems choose $(4, 4)$ and cost £8. The fixed input is already at the right level.

Fixed and quasi-fixed costs

A **fixed cost** F is incurred even if no output is produced. Example: a bakery must pay its contracted rent even if it produces nothing.

A **quasi-fixed cost** Q is independent of the level of output as long as that level is positive. Example: a bakery requires a cleaning service, regardless of how much it produces, but only when output is positive.

We can summarise this as follows:

$$C(y) = \begin{cases} F, & y = 0, \\ F + Q + v(y), & y > 0, \end{cases}$$

where $v(y)$ is the cost of variable inputs.

Fixed and quasi-fixed costs

A **fixed cost** F is incurred even if no output is produced. Example: a bakery must pay its contracted rent even if it produces nothing.

A **quasi-fixed cost** Q is independent of the level of output as long as that level is positive. Example: a bakery requires a cleaning service, regardless of how much it produces, but only when output is positive.

We can summarise this as follows:

$$C(y) = \begin{cases} F, & y = 0, \\ F + Q + v(y), & y > 0, \end{cases}$$

where $v(y)$ is the cost of variable inputs.

In the long run, there are no fixed costs by definition, since all factors can adjust. There may still be quasi-fixed costs.

Sunk costs and recoverable expenditure

A **sunk cost** is a cost that has already been incurred and cannot be recovered. It cannot be affected by the firm's current decision.

Suppose the bakery has just paid £1,000 for equipment that it could immediately resell for £700. Of the purchase expenditure, £300 is sunk.

Next lecture: Cost Curves. How do total, average and marginal costs vary as the output changes?