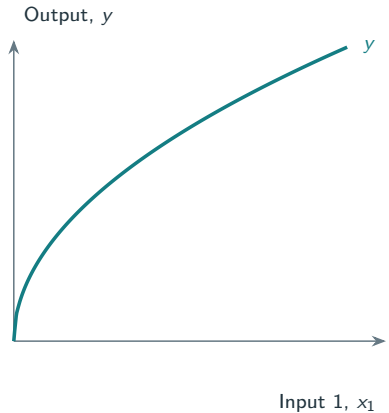
LECTURE 8

# Profit Maximisation

5SSPP217 – Intermediate Microeconomics

# Recap: production and marginal products

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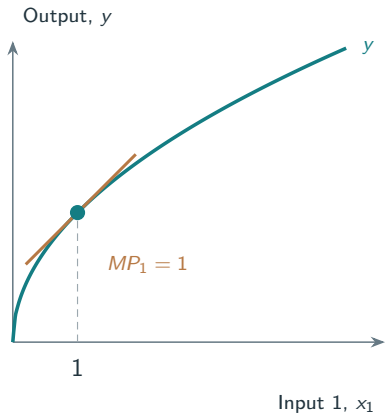


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$$MP_1 = \frac{\partial f}{\partial x_1}.$$



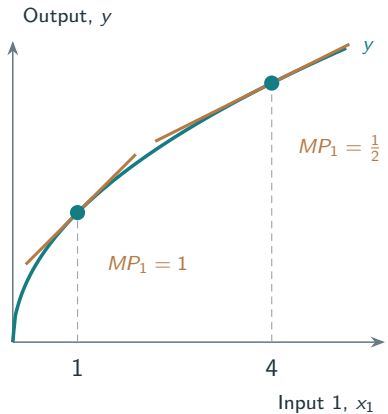
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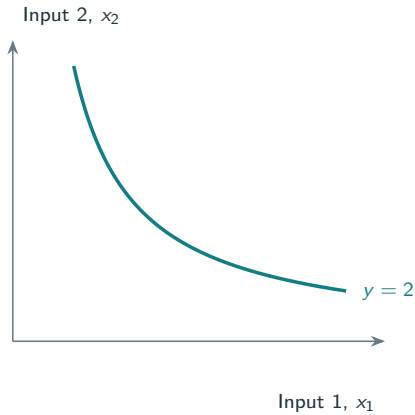
With **diminishing marginal product**, $MP_1$ falls as $x_1$ rises.



# Recap: isoquants and technology

An **isoquant** shows all input bundles with the same maximum output:

$$f(x_1, x_2) = \bar{y}.$$

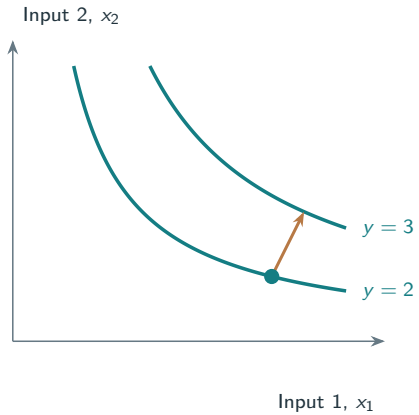


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**Monotonicity**: increasing either input cannot reduce maximum output.



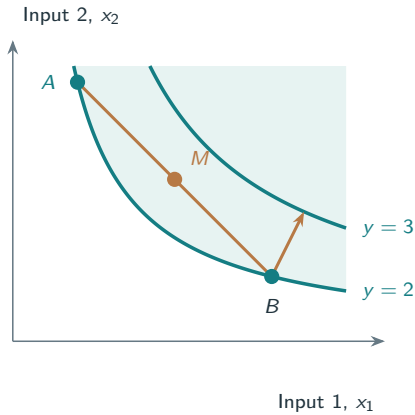
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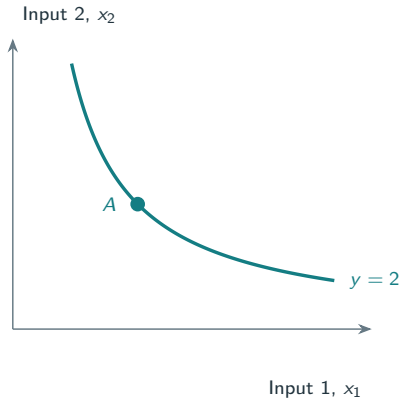
**Monotonicity**: increasing either input cannot reduce maximum output.

**Convexity**: any weighted average of bundles that can produce $\bar{y}$ can also produce at least $\bar{y}$.



# Recap: the technical rate of substitution

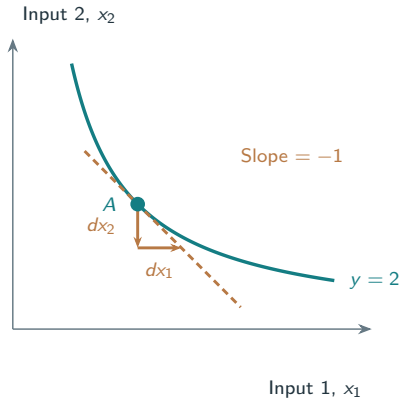
The **technical rate of substitution** measures by how much input 2 can fall if input 1 increases and output must remain fixed.



# Recap: the technical rate of substitution

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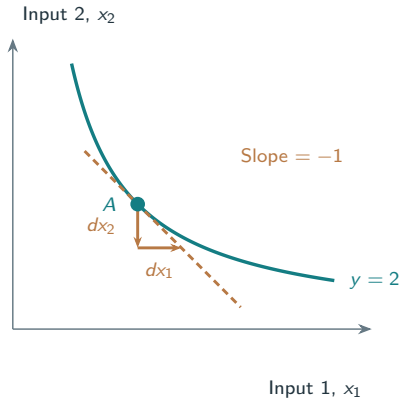
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With $\partial f/\partial x_2 > 0$, rearranging gives

$$TRS = \left.\frac{dx_2}{dx_1}\right|_y = -\frac{\partial f/\partial x_1}{\partial f/\partial x_2}.$$



# Recap: returns to scale

**Returns to scale** describe how output changes when all inputs increase in the same proportion.

Suppose we multiply all inputs by $t > 1$ and compare output with $tf(x_1, x_2)$:

| | |
|---|---|
| **Constant returns** | $f(tx_1, tx_2) = tf(x_1, x_2)$ |
| **Increasing returns** | $f(tx_1, tx_2) > tf(x_1, x_2)$ |
| **Decreasing returns** | $f(tx_1, tx_2) < tf(x_1, x_2)$ |

# Today: profit maximisation

Last lecture, we looked at the constraints faced by the firm.

Today, we look at how firms make choices given those constraints.

We will assume that firms choose the production plan that maximises profits, taking, for now, the prices of inputs and outputs as given.

# The firm's objective

# Profits and opportunity costs

**Profit** is defined as total revenue minus total cost.

Suppose a firm sells output $y$ at price $p$, using labour $L$ at wage rate $w$ and capital services $K$ at rental rate $r$.

The **rental rate** is capital's equivalent of the wage rate: the price per unit of capital services.

Profit $\pi$ is then given by:

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When calculating profit, we need to consider all costs even when there is no actual cash payment, e.g., the time and capital of the owner of a small business.

# Calculating economic profit

Suppose a bakery has weekly revenue of £1,000 and pays £600 for inputs. Its owner could earn £250 working elsewhere and let the premises for £100.

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$$\pi = 1{,}000 - 600 - 250 - 100 = £50.$$

The unpaid work and owned premises still have an opportunity cost: the earnings forgone by using them in the bakery.

The inclusion of opportunity costs makes an economist's definition of profit different from that of accountants.

# Short-run choices

# Fixed and variable factors

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For example, a bakery can easily change the amount of labour employed while its oven capacity is fixed. In the long run, it can adjust both.

# The short-run problem

With input 2 fixed at $\bar{x}_2$ and $p, w_1, w_2$ given, the firm chooses $x_1$ to maximise profit:

$$\max_{x_1 \geq 0} \ \pi(x_1) = pf(x_1, \bar{x}_2) - w_1 x_1 - w_2 \bar{x}_2.$$

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Differentiating with respect to $x_1$ and setting equal to 0:

$$\frac{\partial \pi}{\partial x_1} = p \frac{\partial f(x_1, \bar{x}_2)}{\partial x_1} - w_1 = 0,$$

$$p \frac{\partial f(x_1^*, \bar{x}_2)}{\partial x_1} = w_1.$$

If **value of the marginal product**, $p\,\partial f/\partial x_1$, exceeds $w_1$, increasing $x_1$ raises profit; if it is below $w_1$, reducing $x_1$ raises profit.

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If $f(x_1, \bar{x}_2)$ is concave in $x_1$, this interior solution gives an optimal level of $x_1$; otherwise, check other stationary points and boundaries!

# Solving the short-run example

Let $f(x_1, x_2) = \sqrt{x_1 x_2}$ and $\bar{x}_2 = 4$. Then $f(x_1, 4) = 2\sqrt{x_1}$ and profit is $\pi(x_1) = 2p\sqrt{x_1} - w_1 x_1 - 4w_2$.

The interior condition becomes

$$p\frac{\partial f(x_1, 4)}{\partial x_1} - w_1 = \frac{p}{\sqrt{x_1}} - w_1 = 0,$$

$$\sqrt{x_1^*} = \frac{p}{w_1} \quad \Longrightarrow \quad x_1^* = \left(\frac{p}{w_1}\right)^2.$$

# Solving the short-run example

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Substitute back into output and profit:

$$y^* = 2\frac{p}{w_1}, \qquad \pi^* = \frac{p^2}{w_1} - 4w_2.$$

For $p = 2$, $w_1 = 1$, $w_2 = \frac{1}{2}$: $x_1^* = 4$, $y^* = 4$, $\pi^* = 2$.

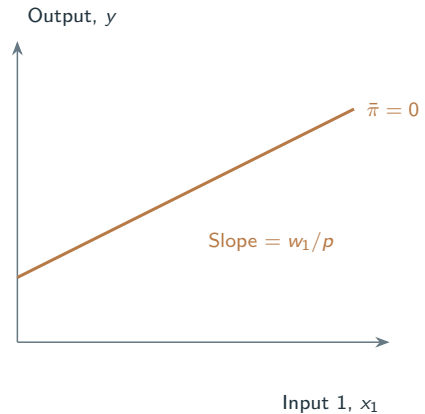
# Isoprofit lines

We can use the profit equation to derive the **isoprofit line**: the combinations of input 1 and output that yield the same profit, $\bar{\pi}$:

$$\bar{\pi} = py - w_1 x_1 - w_2 \bar{x}_2,$$

$$y = \frac{\bar{\pi} + w_2 \bar{x}_2}{p} + \frac{w_1}{p} x_1.$$

One extra unit of $x_1$ costs $w_1$. To keep profit unchanged, output must rise enough to generate $w_1$ in extra revenue: $p\, dy/dx_1 = w_1$, so $dy/dx_1 = w_1/p$.



# Isoprofit lines

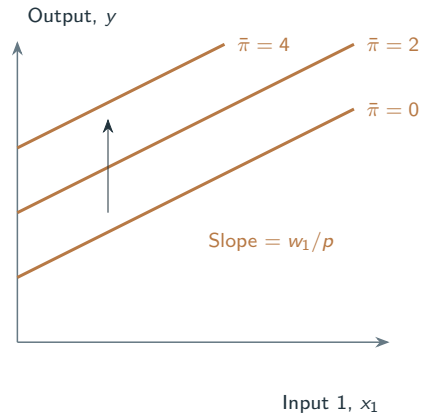
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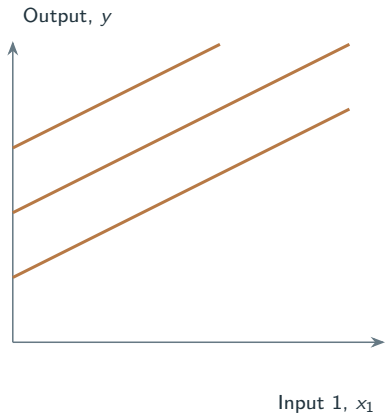
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Higher parallel lines represent higher profit.



# The highest feasible isoprofit line

The firm wants to earn the highest possible profit.

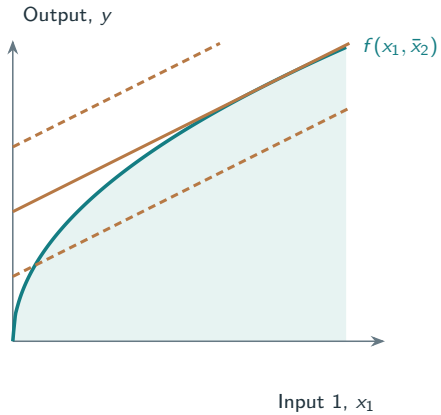


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What constrains the firm? **Technology!**

Output cannot exceed $f(x_1, \bar{x}_2)$.



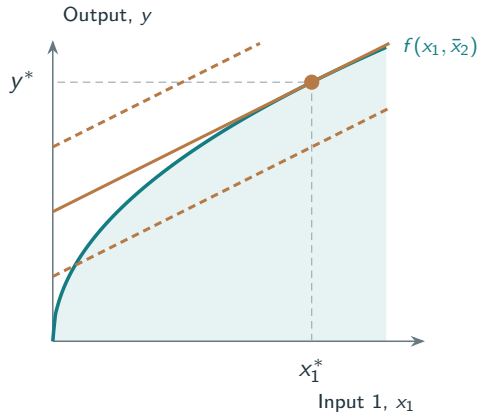
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The firm chooses $x_1$ to reach the **highest attainable isoprofit line**, at $(x_1^*, y^*)$.



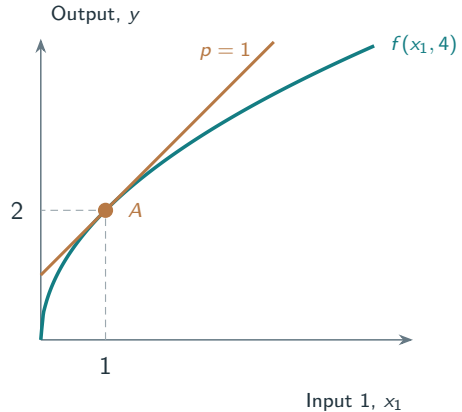
# Comparative statics: a higher output price

Recall that the slope of the isoprofit line is given by:

$$\left.\frac{dy}{dx_1}\right|_{\pi=\bar{\pi}} = \frac{w_1}{p}.$$

Using $f(x_1, x_2) = \sqrt{x_1 x_2}$, hold $\bar{x}_2 = 4$, $w_1 = 1$ and $w_2 = \frac{1}{2}$ fixed. Raise the output price from $p = 1$ to $p = 2$.

The isoprofit slope $w_1/p$ falls from 1 to $\frac{1}{2}$.



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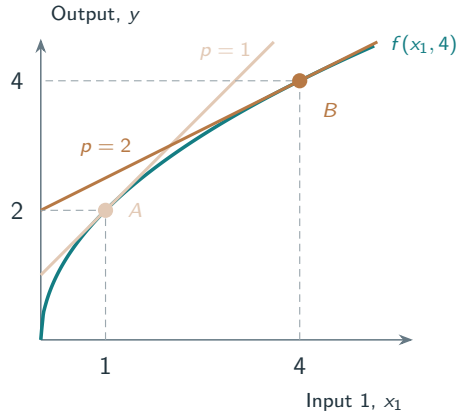
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The optimal plan moves from $A$ to $B$.

Extra output is now more valuable, so the firm uses more of input 1 and produces more.



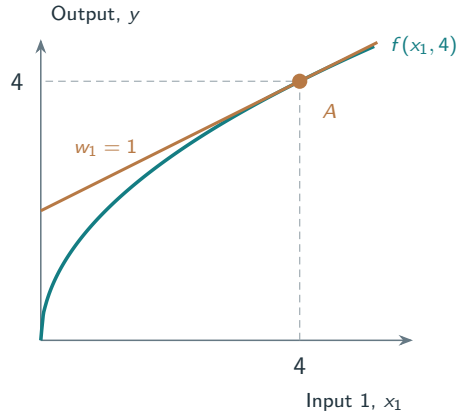
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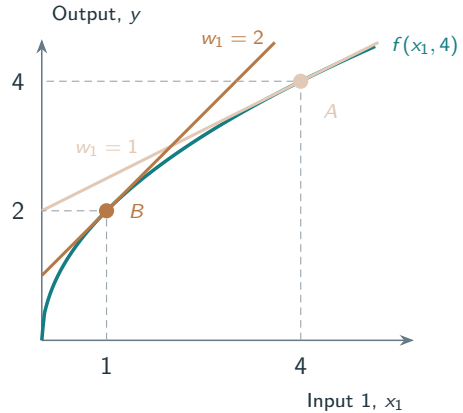
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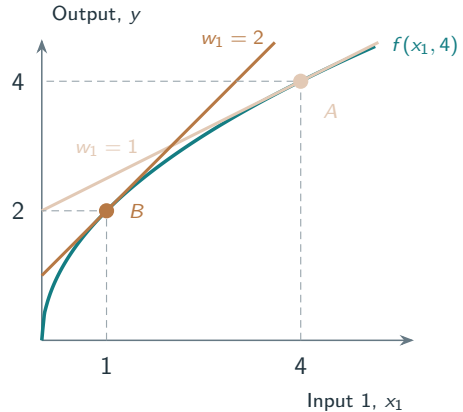
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The isoprofit slope rises from $\frac{1}{2}$ to 1.

The optimal plan moves from $A$ to $B$.

Input 1 is now more expensive, so the firm uses less of it and produces less.




The firm's choice depends on the **relative price** $w_1/p$: a higher input price has the same effect as a lower output price.

# Long-run choices

# Choosing both inputs

In the **long run**, all inputs can vary: the firm maximises profit by choosing both $x_1$ and $x_2$.

$$\max_{x_1, x_2 \geq 0} \pi(x_1, x_2) = pf(x_1, x_2) - w_1 x_1 - w_2 x_2.$$

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If there is an interior optimum, neither input can raise profit at the margin:

$$\frac{\partial \pi}{\partial x_1} = p\frac{\partial f}{\partial x_1} - w_1 = 0 \quad \Longrightarrow \quad p\frac{\partial f}{\partial x_1} = w_1,$$

$$\frac{\partial \pi}{\partial x_2} = p\frac{\partial f}{\partial x_2} - w_2 = 0 \quad \Longrightarrow \quad p\frac{\partial f}{\partial x_2} = w_2.$$

When there is an interior optimum, solving these conditions gives the **factor demand functions** $x_1^*(p, w_1, w_2)$ and $x_2^*(p, w_1, w_2)$: the optimal input quantities given prices.

# A Cobb–Douglas example with decreasing returns

Suppose the production function is given by

$$f(L, K) = L^{1/3} K^{1/3}.$$

The firm solves

$$\max_{L,K \geq 0} \ \pi(L, K) = pL^{1/3} K^{1/3} - wL - rK.$$

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At an interior optimum, the first-order conditions are:

$$p \frac{\partial f}{\partial L} = \frac{p}{3} L^{-2/3} K^{1/3} = w, \qquad p \frac{\partial f}{\partial K} = \frac{p}{3} L^{1/3} K^{-2/3} = r.$$

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Multiply by $L$ and $K$, respectively, and use $y = L^{1/3} K^{1/3}$:

$$\frac{py}{3} = wL, \qquad \frac{py}{3} = rK, \qquad \frac{py}{3} = \pi.$$

# Solving for output and inputs

We can insert the optimal levels of $L = py/(3w)$ and $K = py/(3r)$ into the production function to find the optimal level of output:

$$y = \left(\frac{py}{3w}\right)^{1/3} \left(\frac{py}{3r}\right)^{1/3},$$

$$y^3 = \frac{p^2 y^2}{9wr} \quad \Longrightarrow \quad y^* = \frac{p^2}{9wr} \quad (y > 0).$$

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Then

$$L^* = \frac{p^3}{27w^2r}, \qquad K^* = \frac{p^3}{27wr^2}.$$

For $p = 3$, $w = r = 1$: $L^* = K^* = y^* = 1$, and $\pi^* = 3 - 1 - 1 = 1$.

# Returns to scale and profit

# Constant returns to scale

Suppose that a firm produces output with constant returns to scale. Can it make a positive profit when it produces at the profit-maximising level?

At fixed output and input prices, doubling all inputs doubles output, revenue and costs, and therefore profit.

But this means that the firm's original choice was not profit-maximising!

Negative profit cannot be optimal either: the firm can earn a zero profit by choosing no inputs and no output.

Therefore, if a finite profit maximum exists under constant returns to scale, it must be zero.