

5SSPP217 Microeconomics
Solutions to Week 1 exercises
Maths refresher

1. Profit maximisation.

- (a) With $q \geq 0$, revenue is $200q$, so profit is

$$\pi(q) = 200q - (100 + 160q + 0.5q^2) = 40q - 0.5q^2 - 100.$$

The first-order condition is

$$\pi'(q) = 40 - q = 0, \quad \text{equivalently} \quad p = MC(q) = 160 + q.$$

Hence $\boxed{q^* = 40}$. Since $\pi''(q) = -1 < 0$, this is the unique profit-maximising output. Profit is

$$\pi(40) = 200(40) - [100 + 160(40) + 0.5(40)^2] = \boxed{\mathcal{L}700}.$$

- (b) With fixed cost equal to 1000, profit when operating becomes

$$\tilde{\pi}(q) = 40q - 0.5q^2 - 1000.$$

Marginal cost and the first-order condition are unchanged, so $q^* = 40$ remains the optimal output *conditional on operating*, with profit

$$\tilde{\pi}(40) = 800 - 1000 = \boxed{-\mathcal{L}200}.$$

The firm can avoid the fixed cost by exiting the market. Exit gives zero profit, which is better than operating at a loss of 200. The firm therefore exits, with

$$\boxed{q^* = 0, \quad \pi^* = 0}.$$

2. Level curves.

Use the full real domain of each function. For $\sqrt{x} + y$ this means $x \geq 0$ and $y \in \mathbb{R}$. If only non-negative x and y are intended, retain the portions of the curves in the first quadrant.

- (a) For $f(x, y) = x + y$, the k -level curve is

$$y = k - x, \quad k \in \mathbb{R}.$$

These are parallel straight lines with slope -1 , intersecting the axes at $(k, 0)$ and $(0, k)$.

- (b) For $f(x, y) = xy$, the level curve satisfies $xy = k$. If $k \neq 0$, it can be written as

$$y = \frac{k}{x}, \quad x \neq 0.$$

These are rectangular hyperbolas with the axes as asymptotes. For $k > 0$ their branches lie in quadrants I and III; for $k < 0$ they lie in quadrants II and IV. For $k = 0$, the level set is *both coordinate axes*: $x = 0$ or $y = 0$.

- (c) For $f(x, y) = \sqrt{x} + y$, the k -level curve is

$$y = k - \sqrt{x}, \quad x \geq 0, \quad k \in \mathbb{R}.$$

Its vertical intercept is $(0, k)$. A horizontal intercept exists only for $k \geq 0$, when it is $(k^2, 0)$. For $x > 0$,

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{x}} < 0, \quad \frac{d^2y}{dx^2} = \frac{1}{4x^{3/2}} > 0.$$

The curves are decreasing and convex, with a vertical tangent at $x = 0$. In the first quadrant, $k \geq 0$ and $0 \leq x \leq k^2$.

- (d) For $f(x, y) = x^2 + y^2$, the level curve satisfies

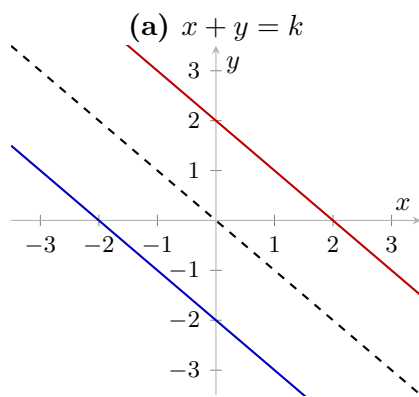
$$x^2 + y^2 = k.$$

For $k > 0$, it is a *circle* centred at the origin with radius $\sqrt{k}$; equivalently,

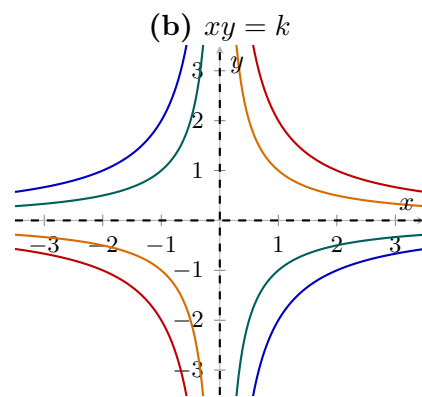
$$y = \pm\sqrt{k - x^2}, \quad |x| \leq \sqrt{k}.$$

Its intercepts are $(\pm\sqrt{k}, 0)$ and $(0, \pm\sqrt{k})$. For $k = 0$, the level set is just $(0, 0)$; for $k < 0$, it is empty. In the first quadrant, the $k > 0$ curves are quarter-circles.

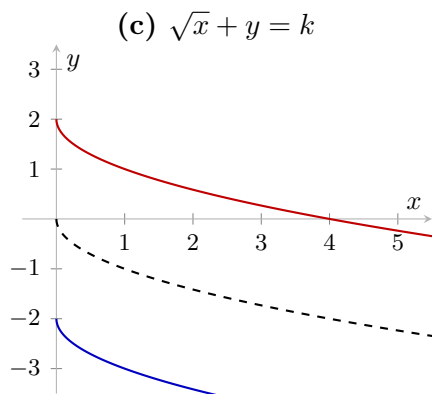
Level-curve maps for Question 2



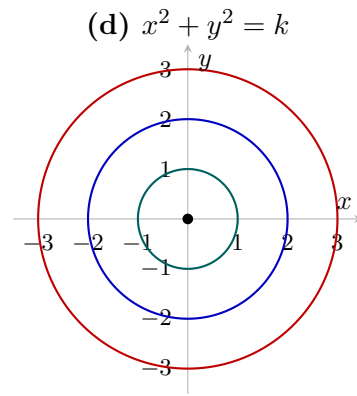
— $k = -2$ --- $k = 0$ — $k = 2$



— $k = -2$ — $k = -1$ --- $k = 0$
— $k = 1$ — $k = 2$



— $k = -2$ --- $k = 0$ — $k = 2$



• $k = 0$ — $k = 1$ — $k = 4$ — $k = 9$

In (b), the dashed axes together form the $k = 0$ level set. In (d), the $k = 0$ level set is the dot at the origin.

3. Utility maximisation.

Assume consumption is non-negative: $x_1, x_2 \geq 0$. With $a, b > 0$, $p_1, p_2 > 0$ and $m > 0$, the Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^a x_2^b - \lambda(p_1 x_1 + p_2 x_2 - m).$$

At an interior optimum, the corresponding first-order conditions are

$$\begin{aligned} a x_1^{a-1} x_2^b - \lambda p_1 &= 0, \\ b x_1^a x_2^{b-1} - \lambda p_2 &= 0, \\ -p_1 x_1 - p_2 x_2 + m &= 0. \end{aligned}$$

Dividing the first condition by the second gives

$$\frac{a}{b} \frac{x_2}{x_1} = \frac{p_1}{p_2}, \quad \text{so} \quad p_2 x_2 = \frac{b}{a} p_1 x_1.$$

Substituting into the budget constraint yields

$$p_1 x_1 + \frac{b}{a} p_1 x_1 = m, \quad p_1 x_1 = \frac{a}{a+b} m.$$

Therefore the consumer's optimal demands are

$$\boxed{x_1^* = \frac{a}{a+b} \frac{m}{p_1}}, \quad \boxed{x_2^* = \frac{b}{a+b} \frac{m}{p_2}}.$$

The consumer spends the shares $a/(a+b)$ and $b/(a+b)$ of income on goods 1 and 2, respectively.

To verify that these first-order conditions identify the global maximum, observe that either endpoint of the budget line gives utility zero, whereas every feasible bundle with both goods strictly positive gives positive utility. Thus an optimum is interior. On the positive domain, maximising U is equivalent to maximising

$$\log U = a \log x_1 + b \log x_2.$$

This function has Hessian

$$\begin{pmatrix} -a/x_1^2 & 0 \\ 0 & -b/x_2^2 \end{pmatrix},$$

which is negative definite. It is strictly concave, so the solution is the unique global maximum on the budget line. No restriction such as $a+b \leq 1$ is required.

4. Cost minimisation.

- (a) Assume $w, r > 0$ and $K, L \geq 0$. The firm minimises $wL + rK$ subject to $(\sqrt{K} + \sqrt{L})^2 = 1000$. Its Lagrangian is

$$\mathcal{L}(L, K, \lambda) = wL + rK - \lambda[(\sqrt{K} + \sqrt{L})^2 - 1000].$$

The interior first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial L} &= w - \lambda L^{-1/2}(\sqrt{K} + \sqrt{L}) = 0, \\ \frac{\partial \mathcal{L}}{\partial K} &= r - \lambda K^{-1/2}(\sqrt{K} + \sqrt{L}) = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 1000 - (\sqrt{K} + \sqrt{L})^2 = 0.\end{aligned}$$

Multiplying the last equation by -1 gives the output constraint in the form used in the question.

- (b) Dividing the first two conditions gives

$$\frac{w}{r} = \frac{L^{-1/2}}{K^{-1/2}} = \sqrt{\frac{K}{L}}, \quad \sqrt{K} = \frac{w}{r}\sqrt{L}.$$

Substitution into the output constraint gives

$$\left(\frac{w}{r}\sqrt{L} + \sqrt{L}\right)^2 = 1000, \quad L\left(1 + \frac{w}{r}\right)^2 = 1000.$$

Hence the *conditional* factor demands are

$$L^*(w, r) = \frac{1000}{\left(1 + \frac{w}{r}\right)^2} = \frac{1000r^2}{(w + r)^2},$$

$$K^*(w, r) = \frac{1000}{\left(1 + \frac{r}{w}\right)^2} = \frac{1000w^2}{(w + r)^2}.$$

They are conditional on the output target of 1000 cars.

To verify global cost minimisation, put $u = \sqrt{L}$ and $v = \sqrt{K}$. Then $u + v = \sqrt{1000}$, and cost becomes

$$c(u) = wu^2 + r(\sqrt{1000} - u)^2, \quad 0 \leq u \leq \sqrt{1000}.$$

Since $c''(u) = 2(w + r) > 0$, cost is strictly convex, and

$$c'(u) = 2wu - 2r(\sqrt{1000} - u) = 0 \implies u^* = \frac{r\sqrt{1000}}{w + r}.$$

This lies strictly inside the feasible interval and reproduces the demands above. Thus they give the unique global cost minimum, including comparison with the two corner bundles. As a check,

$$\sqrt{K^*} + \sqrt{L^*} = \sqrt{1000}, \quad c^* = wL^* + rK^* = \frac{1000wr}{w + r}.$$

5. Marginal products and homogeneity.

For $F(K, L) = AK^\alpha L^\beta$, take $A, \alpha, \beta > 0$ and evaluate derivatives at $K, L > 0$.

(a) The marginal products are

$$MP_L = \frac{\partial F}{\partial L} = A\beta K^\alpha L^{\beta-1} = A\beta \frac{K^\alpha}{L^{1-\beta}} > 0,$$

$$MP_K = \frac{\partial F}{\partial K} = A\alpha K^{\alpha-1} L^\beta = A\alpha \frac{L^\beta}{K^{1-\alpha}} > 0.$$

Both are positive for all positive exponents. To determine whether they are diminishing, differentiate each marginal product with respect to its *own input*, holding the other input fixed:

$$\frac{\partial MP_L}{\partial L} = A\beta(\beta-1)K^\alpha L^{\beta-2} = -(1-\beta)\beta A \frac{K^\alpha}{L^{2-\beta}},$$

$$\frac{\partial MP_K}{\partial K} = A\alpha(\alpha-1)K^{\alpha-2} L^\beta = -(1-\alpha)\alpha A \frac{L^\beta}{K^{2-\alpha}}.$$

Therefore

$$MP_L \text{ is strictly diminishing} \iff 0 < \beta < 1,$$

$$MP_K \text{ is strictly diminishing} \iff 0 < \alpha < 1.$$

Both marginal products are strictly diminishing if and only if

$$\boxed{0 < \alpha < 1 \text{ and } 0 < \beta < 1}.$$

If the relevant exponent equals 1, the marginal product is constant in its own input; if it exceeds 1, that marginal product is increasing.

(b) For any $t > 0$,

$$\begin{aligned} F(tK, tL) &= A(tK)^\alpha (tL)^\beta \\ &= t^{\alpha+\beta} AK^\alpha L^\beta \\ &= t^{\alpha+\beta} F(K, L). \end{aligned}$$

Hence the function is homogeneous of degree $\boxed{\alpha + \beta}$.

As an economic interpretation, returns to scale are decreasing if $\alpha + \beta < 1$, constant if $\alpha + \beta = 1$, and increasing if $\alpha + \beta > 1$. Returns to scale concern changing *both* inputs together; diminishing marginal products concern changing *one* input while holding the other fixed. Diminishing marginal products therefore do not, by themselves, imply decreasing returns to scale.